Lagrangian

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The Lagrangian of a dynamical system is a function of generalised co-ordinates (q_k), generalised velocity ($\dot{q}_k$) and time (t). i.e, $L = f(q_k, \dot{q}_k, t)$

Again Lagrangian is nothing but the difference of kinetic energy (T) and Potential energy (V) of a system.

i.e, $\boxed{L = T - V}$

Lagrange's equation of motion :-

According to transformation equations, the position vectors of the particles $r_1, r_2 \ldots r_i$ can be expressed as the function of generalised co-ordinates $q_1, q_2 \ldots q_k$ and time t. i.e, $\vec{r}_i = r_i(q_1, q_2 \ldots q_k, t)$ — (1)

for any infinitesimal virtual displacement, time is const.

Hence,

$$\partial\vec{r}_i = \frac{\partial\vec{r}_i}{\partial q_1} dq_1 + \frac{\partial\vec{r}_i}{\partial q_2} dq_2 + \ldots + \frac{\partial\vec{r}_i}{\partial q_k} dq_k$$

$$\partial\vec{r}_i = \sum_k \frac{\partial\vec{r}_i}{\partial q_k} dq_k \quad \text{— (2)}$$

From D'Alembert's Principle,

$$\sum_i (\vec{F}_i - \dot{\vec{P}}_i)\cdot\partial\vec{r}_i = 0 \quad \text{— (3)}$$

Now, $\underline{\sum_i \vec{F}_i \cdot \partial\vec{r}_i}$ [Putting the value of $\partial\vec{r}_i$ in eq^n (3)]

$$\Rightarrow \sum_i \vec{F}_i \cdot \sum_k \frac{\partial\vec{r}_i}{\partial q_k} dq_k$$

(9)

$$\Rightarrow \sum_i \sum_k \left(\vec{F_i} \cdot \frac{\partial \vec{r_i}}{\partial q_k} \right) \cancel{\delta q_k}\, dq_k$$

$$= \sum_k Q_k \, dq_k \longrightarrow (4)$$

where, $Q_k = \sum_i \vec{F_i} \cdot \frac{\partial \vec{r_i}}{\partial q_k}$ is the generalised force.

For conservative system, the force is derived from potential energy i.e, $\vec{F_i} = -\vec{\nabla} V_i$

Thus generalised force, $Q_k = \sum_i \vec{F_i} \cdot \frac{\partial \vec{r_i}}{\partial q_k}$

$$= -\sum_i \vec{\nabla} V_i \cdot \frac{\partial \vec{r_i}}{\partial q_k}$$

$$= -\sum_i \frac{\partial V_i}{\partial \vec{r_i}} \cdot \frac{\partial \vec{r_i}}{\partial q_k}$$

$$= -\sum_i \frac{\partial V_i}{\partial q_k} = -\frac{\partial}{\partial q_k} \sum_i V_i$$

$$Q_k = -\frac{\partial V}{\partial q_k} \longrightarrow (5)$$

where, $\sum_i V_i = V$, total P.E of the system, Now, we find $\sum_i \dot{\vec{p}_i} \cdot \delta\vec{r_i} = \sum_i m_i \ddot{\vec{r_i}} \cdot \partial\vec{r_i}$ (10)

Putting the value of $\partial\vec{r_i}$

$$\Rightarrow \sum_i (m_i \ddot{\vec{r_i}}) \cdot \sum_k \frac{\partial\vec{r_i}}{\partial q_k} dq_k$$

$$\Rightarrow \sum_i \sum_k \left(m_i \ddot{\vec{r_i}} \cdot \frac{\partial\vec{r_i}}{\partial q_k} \right) dq_k$$

$$\Rightarrow \sum_i \sum_k \left[\frac{d}{dt}\left(m_i \dot{\vec{r_i}} \cdot \frac{\partial\vec{r_i}}{\partial q_k} \right) - m_i \dot{\vec{r_i}} \frac{d}{dt}\left(\frac{\partial\vec{r_i}}{\partial q_k} \right) \right] dq_k \rightarrow (6)$$

NOTE**

$$\frac{d}{dt}\left(\sum m_i \dot{\vec{r_i}} \frac{\partial\vec{r_i}}{\partial q_k} \right) = \sum m_i \ddot{\vec{r_i}} \frac{\partial\vec{r_i}}{\partial q_k} + \sum m_i \dot{\vec{r_i}} \frac{d}{dt}\left(\frac{\partial\vec{r_i}}{\partial q_k} \right)$$

or, $$\frac{d}{dt}\left(\sum m_i \dot{\vec{r_i}} \frac{\partial\vec{r_i}}{\partial q_k} \right) - \sum m_i \dot{\vec{r_i}} \frac{d}{dt}\left(\frac{\partial\vec{r_i}}{\partial q_k} \right) = \sum m_i \ddot{\vec{r_i}} \frac{\partial\vec{r_i}}{\partial q_k}$$

Again. $$\left[\frac{d}{dt}\left(\frac{\partial\vec{r_i}}{\partial q_k} \right) = \frac{\partial}{\partial q_k}\left(\frac{d\vec{r_i}}{dt} \right) = \frac{\partial}{\partial q_k}(\vec{v_i}) \right] \rightarrow (7)$$

From eqn (1) $\vec{r_i} = r_i(q_1, q_2 \ldots q_k, t)$

Diffn Partially w.r.to t.

$$\frac{d\vec{r_i}}{dt} = \frac{\partial\vec{r_i}}{\partial q_1}\frac{dq_1}{dt} + \frac{\partial\vec{r_i}}{\partial q_2}\frac{dq_2}{dt} + \ldots + \frac{\partial\vec{r_i}}{\partial t}\cdot\frac{dt}{dt}$$

$$\Rightarrow \vec{v_i} = \sum_k \frac{\partial\vec{r_i}}{\partial q_k} \dot{q_k} + \frac{\partial\vec{r_i}}{\partial t}$$

Again differentiating with r. to. $\dot{q_k}$

we get, $$\frac{\partial\vec{v_i}}{\partial \dot{q_k}} = \frac{\partial\vec{r_i}}{\partial q_k} \rightarrow (8)$$

Putting the ~~value of~~ eqn (7) & (8) in eqn (6)

$$\sum_i \sum_k \left[\frac{d}{dt}\left(m_i \vec{v_i} \cdot \frac{\partial\vec{v_i}}{\partial \dot{q_k}} \right) - m_i \vec{v_i} \frac{\partial}{\partial q_k} \vec{v_i} \right] dq_k$$

(11)

$$\Rightarrow \sum_k \left[\frac{d}{dt}\left\{\frac{\partial}{\partial \dot{q}_k}\left(\sum_i \frac{1}{2} m_i v_i^2\right)\right\} - \frac{\partial}{\partial q_k}\left(\sum_i \frac{1}{2} m_i v_i^2\right)\right] \delta q_k$$

$$\Rightarrow \sum_k \left[\frac{d}{dt}\left\{\frac{\partial T}{\partial \dot{q}_k}\right\} - \frac{\partial}{\partial q_k} T\right] \delta q_k \longrightarrow (9)$$

where $T = \sum_i \frac{1}{2} m_i v_i^2$ is total kinetic energy of the system.

Now, putting eq^n (4) & (9) in eq^n (3)

we get. $\sum_i (\vec{F_i} - \dot{\vec{P_i}}) \cdot \partial \vec{r_i} = 0$

$$\Rightarrow \sum_k \left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k}\right] \delta q_k = \sum_k Q_k \, \delta q_k$$

The above equation holds for all values of K, thus.

$$\boxed{\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k} = Q_K} \rightarrow (10)$$

putting the value of Q_K in equation (10)

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k} = -\frac{\partial V}{\partial q_k}$$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial}{\partial q_k}(T - V) = 0$$

If V is not a function of generalised velocity. then we may write, $\Rightarrow \frac{\partial V}{\partial \dot{q}_k} = 0$

So, Hence.

$$\frac{d}{dt}\left[\frac{\partial}{\partial \dot{q}_k}(T - V)\right] - \frac{\partial}{\partial q_k}(T - V) = 0$$

$$\Rightarrow \boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0}$$

where $L = T - V$

This is the Euler-Lagrangian equation